On supersymmetric solutions in $D = 11$ supergravity with Ricci-flat internal spaces

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Abstract

An introduction to supersymmetric (SUSY) solutions defined on the product of Ricci-flat spaces in $D = 11$ supergravity is presented. The paper contains some background information: (i) decomposition relations for SUSY equations and (ii) 2^{-k}-splitting theorem that explains the appearance of $N = 2^{-k}$ fractional supersymmetries. Examples of $M2$ and $M5$ branes on the product of two Ricci-flat spaces are considered and formulae for (fractional) numbers of unbroken SUSY are obtained.

1 Introduction

In this paper we start "SUSY investigations" of solutions with (intersecting composite non-localized) p-branes from [2]-[13]). These solutions have block-diagonal metrics defined on the product of Ricci-flat spaces $M_0 \times M_1 \times \ldots \times M_n$ and are governed by harmonic functions on M_0. Thus, we are interesting in a subclass of SUSY solutions ¿from [2]-[13]), when certain supergravity theories are considered. Here we consider the case of $D = 11$ supergravity [1].

We note that for flat internal spaces and M_0 the SUSY solutions were considered intensively in numerous publications (see, e.g. [26]-[37] and references therein).

Recently certain SUSY solutions in $D = 10, 11$ supergravities with several internal Ricci-flat internal spaces were considered [14]-[25]. Some of them may be obtained by a simple replacing of flat metrics by Ricci-flat ones. We note that major part of these exact solutions (regardless to SUSY properties) are not new ones but are special cases of those obtained before (see [2]-[13], and references therein). (For example, the magnetic 5-brane solution from [16] with $N = 1/4$ SUSY is a special case of solutions ¿from [2, 3, 4] etc).

Here we suggest a more general approach for investigation of the solutions to "Killing-like" SUSY equations in the backgrounds with block-diagonal metric defined on the product of Ricci-flat spaces $M_0 \times M_1 \times \ldots \times M_n$ with arbitrary (though restricted) n. The 4-form is (tacitly) assumed to be a sum of p-brane monoms of magnetic and electric types.

The paper contains some background treatment: (**i**) decomposition relation for spin connection and matrix-valued covector fields that appear in the SUSY equation; (**ii**) 2^{-k}-splitting theorem for k commuting linear operators.

Here we consider the simplest examples of $M2$ and $M5$ branes defined on the product of two Ricci-flat spaces and obtain formulae for fractional number of SUSY. We also consider the simplest $M2 \cap M5$-configuration (defined on the product of flat spaces) to show how the 2^{-k}-splitting theorem works.

2 Basic notations

Now we describe the basic notations (in arbitrary dimension D).

2.1 Product of manifolds

Here we consider the manifolds

$$M = M_0 \times M_1 \times \ldots \times M_n, \tag{2.1}$$

with the metrics

$$g = e^{2\gamma(x)} \hat{g}^0 + \sum_{i=1}^{n} e^{2\phi^i(x)} \hat{g}^i, \tag{2.2}$$

where $g^0 = g^0_{\mu\nu}(x) dx^\mu \otimes dx^\nu$ is a metric on the manifold M_0 and $g^i = g^i_{m_i n_i}(y_i) dy_i^{m_i} \otimes dy_i^{n_i}$ is a metric on M_i, $i = 1, \ldots, n$.

Here $\hat{g}^i = p_i^* g^i$ is the pullback of the metric g^i to the manifold M by the canonical projection: $p_i : M \to M_i$, $i = 0, \ldots, n$.

The functions $\gamma, \phi^i : M_0 \to \mathbf{R}$ are smooth. We denote $d_\nu = \dim M_\nu$; $\nu = 0, \ldots, n$; $D = \sum_{\nu=0}^{n} d_\nu$. We put any manifold M_ν, $\nu = 0, \ldots, n$, to be oriented and connected. Then the volume d_i-form

$$\tau_i \equiv \sqrt{|g^i(y_i)|}\ dy_i^1 \wedge \ldots \wedge dy_i^{d_i}, \tag{2.3}$$

and signature parameter

$$\varepsilon(i) \equiv \mathrm{sign}(\det(g^i_{m_i n_i})) = \pm 1 \tag{2.4}$$

are correctly defined for all $i = 1, \ldots, n$. Let $\Omega = \Omega(n)$ be a set of all non-empty subsets of $\{1, \ldots, n\}$ ($|\Omega| = 2^n - 1$). For any $I = \{i_1, \ldots, i_k\} \in \Omega$, $i_1 < \ldots < i_k$, we denote

$$\tau(I) \equiv \hat{\tau}_{i_1} \wedge \ldots \wedge \hat{\tau}_{i_k}, \tag{2.5}$$

$$\varepsilon(I) \equiv \varepsilon(i_1) \ldots \varepsilon(i_k), \tag{2.6}$$

$$d(I) \equiv \sum_{i \in I} d_i. \tag{2.7}$$

Here $\hat{\tau}_i = p_i^* \hat{\tau}_i$ is the pullback of the form τ_i to the manifold M by the canonical projection: $p_i : M \to M_i$, $i = 1, \ldots, n$.

2.2 Diagonalization of metric

For the metric $g = g_{MN}(x) dx^M \otimes dx^N$ from (2.2), $M, N = 0, \ldots, D-1$, defined on the manifold (2.1), we define the diagonalizing D-bein $e^A = e^A{}_M dx^M$

$$g_{MN} = \eta_{AB} e^A{}_M e^B{}_N, \qquad \eta_{AB} = \eta^{AB} = \eta_A \delta_{AB}, \tag{2.8}$$

$\eta_A = \pm 1$; $A, B = 0, \ldots, D-1$.

We choose the following frame vectors

$$(e^A{}_M) = \mathrm{diag}(e^{\gamma} e^{(0)a}{}_\mu, e^{\phi^1} e^{(1)a_1}{}_{m_1}, \ldots, e^{\phi^n} e^{(n)a_n}{}_{m_n}), \tag{2.9}$$

where

$$g^0_{\mu\nu} = \eta^{(0)}_{ab} e^{(0)a}{}_\mu e^{(0)b}{}_\nu, \qquad g^i_{m_i n_i} = \eta^{(i)}_{a_i b_i} e^{(i)a_i}{}_{m_i} e^{(i)b_i}{}_{n_i}, \tag{2.10}$$

$i = 1, \ldots, n$, and

$$(\eta_{AB}) = \mathrm{diag}(\eta^{(0)}_{ab}, \eta^{(1)}_{a_1 b_1}, \ldots, \eta^{(n)}_{a_n b_n}). \tag{2.11}$$

For $(e^M{}_A) = (e^A{}_M)^{-1}$ we get

$$(e^M{}_A) = \mathrm{diag}(e^{-\gamma} e^{(0)\mu}{}_a, e^{-\phi^1} e^{(1)m_1}{}_{a_1}, \ldots, e^{-\phi^n} e^{(n)m_n}{}_{a_n}), \tag{2.12}$$

where $(e^{(0)\mu}{}_a) = (e^{(0)a}{}_\mu)^{-1}$, $(e^{(i)m_i}{}_{a_i}) = (e^{(i)a_i}{}_{n_i})^{-1}$, $i = 1, \ldots, n$.

Indices. For indices we also use an alternative numbering: $A = (a, a_1, \ldots, a_n)$, $B = (b, b_1, \ldots, b_n)$, where $a, b = 1_0, \ldots, (d_0)_0$; $a_1, b_1 = 1_1, \ldots, (d_1)_1$; ...; $a_n, b_n = 1_n, \ldots, (d_n)_n$; and $M = (\mu, m_1, \ldots, m_n)$, $N = (\nu, n_1, \ldots, n_n)$, where $\mu, \nu = 1_0, \ldots, (d_0)_0$; $m_1, n_1 = 1_1, \ldots, (d_1)_1$; ...; $m_n, n_n = 1_n, \ldots, (d_n)_n$.

2.3 Gamma-matrices

In what follows $\hat{\Gamma}_A$ are "frame" Γ-matrices satisfying

$$\hat{\Gamma}_A \hat{\Gamma}_B + \hat{\Gamma}_B \hat{\Gamma}_A = 2\eta_{AB} \mathbf{1}, \tag{2.13}$$

$A, B = 0, \ldots, D-1$. Here $\mathbf{1} = \mathbf{1}_D$ is unit $D \times D$ matrix. We also use "world" Γ-matrices

$$\Gamma_M = e^A{}_M \hat{\Gamma}_A, \qquad \Gamma_M \Gamma_N + \Gamma_N \Gamma_M = 2 g_{MN} \mathbf{1}, \tag{2.14}$$

$M, N = 0, \ldots, D-1$, and the matrices with upper indices: $\hat{\Gamma}^A = \eta^{AB} \hat{\Gamma}_B$ and $\Gamma^M = g^{MN} \Gamma_N$.

2.4 Spin connection

Here we use the standard definition for the spin connection

$$\omega^A{}_{BM} = \omega^A{}_{BM}(e, \eta) = e^A{}_N \bigtriangledown_M [g(e, \eta)] e^N{}_B, \tag{2.15}$$

where the covariant derivative $\bigtriangledown_M[g]$ corresponds to the metric $g = g(e, \eta)$ from (2.8). The spinorial covariant derivative reads

$$D_M = \partial_M + \frac{1}{4} \omega_{ABM} \hat{\Gamma}^A \hat{\Gamma}^B, \tag{2.16}$$

where $\omega_{ABM} = \eta_{AA'} \omega^{A'}{}_{BM}$.

The non-zero components of the spin connection (2.15) in the frame (2.9) read

$$\omega^a{}_{b\mu} = \omega^a{}_{b\mu}(e^{(0)}, \eta^{(0)}) - e^{(0)\nu a} \gamma_{,\nu} e^{(0)}_{b\mu} + e^{(0)\nu}{}_b \gamma_{,\nu} e^{(0)a}{}_\mu, \tag{2.17}$$

$$\omega^a{}_{a_i m_j} = -\delta_{ij} e^{\phi^i - \gamma} (e^{(0)a}{}_\nu \bigtriangledown^\nu [g^{(0)}] \phi^i) e^{(i)}_{a_i m_i}, \tag{2.18}$$

$$\omega^{a_i}{}_{a m_j} = \delta_{ij} e^{\phi^i - \gamma} (e^{(0)\nu}{}_a \partial_\nu \phi^i) e^{(i) a_i}{}_{m_i}, \tag{2.19}$$

$$\omega^{a_i}{}_{b_j m_k} = \delta_{ij} \delta_{jk} \omega^{a_i}{}_{b_i m_i}(e^{(i)}, \eta^{(i)}), \tag{2.20}$$

$i, j, k = 1, \ldots, n$, where $\omega^a{}_{b\mu}(e^{(0)}, \eta^{(0)})$ and $\omega^{a_i}{}_{b_i m_i}(e^{(i)}, \eta^{(i)})$, are components of the spin connections corresponding to the metrics from (2.10).

Let

$$A_M \equiv \omega_{ABM} \hat{\Gamma}^A \hat{\Gamma}^B. \tag{2.21}$$

For $A_M = A_M(e, \eta, \hat{\Gamma})$ in the frame (2.9) we get

$$A_\mu = \omega^{(0)}_{ab\mu} \hat{\Gamma}^a \hat{\Gamma}^b + (\Gamma_\mu \Gamma^\nu - \Gamma^\nu \Gamma_\mu) \gamma_{,\nu}, \tag{2.22}$$

$$A_{m_i} = \omega^{(i)}_{a_i b_i m_i} \hat{\Gamma}^{a_i} \hat{\Gamma}^{b_i} + 2 \Gamma_{m_i} \Gamma^\nu \phi^i_{,\nu}, \tag{2.23}$$

where $\omega^{(0)}_{ab\mu} = \omega_{ab\mu}(e^{(0)}, \eta^{(0)})$ and $\omega^{(i)}_{a_i b_i m_i} = \omega_{a_i b_i m_i}(e^{(i)}, \eta^{(i)})$, $i = 1, \ldots, n$.

3 SUSY equations

We consider the $D = 11$ supergravity with the action in the bosonic sector [1]

$$S = \int d^{11}z\sqrt{|g|}\Big\{R[g] - \frac{1}{4!}F^2\Big\} + c_{11}\int A \wedge F \wedge F, \tag{3.1}$$

where $c_{11} =$ const and $F = dA$ is 4-form. Here we consider pure bosonic configurations in $D = 11$ supergravity (with zero fermionic fields) that are solutions to the equations of motion corresponding to the action (3.1).

The number of supersymmetries (SUSY) corresponding to the bosonic background $(e^A_M, A_{M_1M_2M_3})$ is defined by a dimension of the space of solutions to (a set of) linear first-order differential equations (SUSY eqs.)

$$(D_M + B_M)\varepsilon = 0, \tag{3.2}$$

where D_M is covariant spinorial derivative ¿from (2.16), $\varepsilon = \varepsilon(z)$ is 32-component "real" spinor field (see Remark 1 below) and

$$B_M = \frac{1}{144\sqrt{2}}(\Gamma_M\Gamma^N\Gamma^P\Gamma^Q\Gamma^R - 12\delta^N_M\Gamma^P\Gamma^Q\Gamma^R)F_{NPQR}. \tag{3.3}$$

Here $F = dA = \frac{1}{4!}F_{NPQR}dz^N \wedge dz^P \wedge dz^Q \wedge dz^R$, and Γ_M are world Γ-matrices.

Remark 1. More rigorously, $\varepsilon(z) \in \mathbf{R}^{0,32}_{\mathbf{G}} = (\mathbf{G_1})^{32}$, where $\mathbf{G_1}$ is an odd part of the infinite-dimensional Grassmann-Banach algebra (over $\mathbf{R}$) $\mathbf{G} = \mathbf{G}_0 \oplus \mathbf{G}_1$ [38].

Here we consider the decomposition of matrix-valued field B_M on the product manifold (2.1) in the frame (2.9) for electric and magnetic branes.

3.1 $M2$-brane.

Let the 4-form be

$$F = d\Phi \wedge \tau(I) \tag{3.4}$$

where $\Phi = \Phi(x)$, $I = \{i_1, \ldots, i_k\}$, $i_1 < \ldots < i_k$, $d(I) = 3$. The calculations give

$$B_{m_l} = \frac{1}{6\sqrt{2}}s(I)\exp(-\sum_{i\in I} d_i\phi^i)[(1 - 3\delta^l_I)\Gamma_{m_l}\Gamma^\nu\Phi_{,\nu} - 3\delta^l_0\Phi_{,m_l}]\hat{\Gamma}(I), \tag{3.5}$$

where $l = 0, \ldots, n$, $m_0 = \mu$, $s(I) = \mathrm{sign}(\prod_{i\in I}\det(e^{(i)m_i}{}_{a_i}))$ and $\hat{\Gamma}(I) = \hat{\Gamma}^{\bar{1}}\hat{\Gamma}^{\bar{2}}\hat{\Gamma}^{\bar{3}}$ with $(\bar{1}, \bar{2}, \bar{3}) = (1_{i_1}, \ldots, (d_{i_1})_{i_1}, \ldots, 1_{i_k}, \ldots, (d_{i_k})_{i_k})$.

3.2 $M5$-brane.

Let

$$F = (*_0 d\Phi) \wedge \tau(\bar{I}), \tag{3.6}$$

where $*_0$ is the Hodge operator on (M_0, g^0) and $\bar{I} = \{1, \dots, n\} \setminus I = \{j_1, \dots, j_l\}$, $j_1 < \dots < j_l$. It follows from (3.6) that $d_0 + d(\bar{I}) = 5$ and $d(I) = 6$. We get

$$B_{m_l} = \frac{1}{12\sqrt{2}} s(\{0\}) s(\bar{I}) \exp[-(d_0 - 2)\gamma - \sum_{i \in \bar{I}} d_i \phi^i] \times \tag{3.7}$$
$$\times [2\Gamma_{m_l}\Gamma^{\nu}\Phi_{,\nu} - 3\delta_0^l(\Gamma_{m_l}\Gamma^{\nu} - \Gamma^{\nu}\Gamma_{m_l})\Phi_{,\nu} + 6\delta_{\bar{I}}^l \Gamma^{\nu}\Gamma_{m_l})\Phi_{,\nu}] \hat{\Gamma}(\{0\})\hat{\Gamma}(\bar{I}),$$

where $l = 0, \dots, n$; $s(\{0\}) = \mathrm{sign}(\det(e^{(0)\nu}{}_a))$, $\hat{\Gamma}(\{0\}) = \hat{\Gamma}^{1_0} \dots \hat{\Gamma}^{(d_0)_0}$ and $\hat{\Gamma}(\bar{I}) = \hat{\Gamma}^{\bar{1}} \dots \hat{\Gamma}^{\bar{k}}$ with $(\bar{1}, \dots, \bar{k}) = (1_{j_1}, \dots, (d_{j_1})_{j_1}, \dots, 1_{j_l}, \dots, (d_{j_l})_{j_l})$.

3.3 2^{-k}-splitting theorem

In next section some examples of supersymmetric solutions will be considered. In counting the fractional number of supersymmetries the following (2^{-k}-splitting) theorem is used.

Theorem. *Let V be a vector space over $K = \mathbf{R}, \mathbf{C}$; $V \neq \{0\}$. Let $\Gamma_{[i]} : V \to V$, $i = 1, \dots, k$, be a set of linear mappings (operators) satisfying:*

$$\Gamma_{[i]}^2 = \mathrm{id}_V \equiv \mathbf{1}, \qquad \Gamma_{[i]} \circ \Gamma_{[j]} = \Gamma_{[j]} \circ \Gamma_{[i]}, \tag{3.8}$$

$i, j = 1, \dots, k$. Then

$$V = \oplus \sum_{s_1, \dots, s_k = \pm 1} V_{s_1, \dots, s_k}, \tag{3.9}$$

where

$$V_{s_1, \dots, s_k} \equiv \{x \in V | \Gamma_{[i]} x = s_i x, i = 1, \dots, k\}, \tag{3.10}$$

are subspaces of V, $s_1, \dots, s_k = \pm 1$. Moreover, if there exists a set of linear bijective mappings $A_{[i]} : V \to V$, $i = 1, \dots, k$, satisfying

$$\begin{aligned} A_{[i]} \circ \Gamma_{[i]} &= -\Gamma_{[i]} \circ A_{[i]}, \\ A_{[i]} \circ \Gamma_{[j]} &= \Gamma_{[j]} \circ A_{[i]}, \quad i \neq j, \end{aligned} \tag{3.11}$$

$i, j = 1, \dots, k$, then all subspaces $V_{s_1, \dots, s_k}$ are mutually isomorphic and for finite-dimensional V

$$\mathrm{dim} V_{s_1, \dots, s_k} = 2^{-k} \mathrm{dim} V, \tag{3.12}$$

$s_1, \dots, s_k = \pm 1$.

Proof. Let us introduce a set of projector operators

$$P_s^{[i]} = \frac{1}{2}(\mathbf{1} + s\Gamma_{[i]}), \tag{3.13}$$

$i = 1, \dots, k$; $s = \pm 1$, satisfying

$$(P_s^{[i]})^2 = P_s^{[i]}, \tag{3.14}$$
$$P_s^{[i]} + P_{-s}^{[i]} = \mathbf{1}, \tag{3.15}$$
$$P_s^{[i]} \circ P_{-s}^{[i]} = \mathbf{0}, \tag{3.16}$$
$$P_s^{[i]} \circ P_{s'}^{[j]} = P_{s}^{[j]} \circ P_{s'}^{[i]}, \tag{3.17}$$

for all $i, j = 1, \ldots, k$; $s, s' = \pm 1$. $\mathbf{0}$ is the zero-operator. The relation (3.15) implies

$$(P^{[1]}_{+1} + P^{[1]}_{-1}) \circ \ldots \circ (P^{[k]}_{+1} + P^{[k]}_{-1}) = \sum_{s_1,\ldots,s_k=\pm 1} P^{[1]}_{s_1} \circ \ldots \circ P^{[k]}_{s_k}. \tag{3.18}$$

By definition

$$V_{s_1,\ldots,s_k} = \mathrm{Ker} P^{[1]}_{-s_1} \cap \ldots \cap \mathrm{Ker} P^{[k]}_{-s_k}. \tag{3.19}$$

It may be verified using (3.13)-(3.18) that

$$V_{s_1,\ldots,s_k} = (P^{[1]}_{s_1} \circ \ldots \circ P^{[k]}_{s_k}) V \tag{3.20}$$

and the decomposition (3.9) holds. ¿From (3.11) and (3.13) we get

$$A_{[i]} \circ P^{[i]}_{s} = P^{[i]}_{-s} \circ A_{[i]}, \tag{3.21}$$

$$A_{[i]} \circ P^{[j]}_{s} = P^{[j]}_{s} \circ A_{[i]}. \qquad i \neq j, \tag{3.22}$$

$i, j = 1, \ldots, k$; $s = \pm 1$. Let us introduce linear functions

$$\begin{aligned} A_{s_1,\ldots,s_k} : V_{s_1,\ldots,s_i,\ldots,s_k} &\to V_{s_1,\ldots,-s_i,\ldots,s_k} \\ v &\mapsto A_{[i]} v. \end{aligned} \tag{3.23}$$

It follows from (3.20), (3.21) and (3.22) that these functions are correctly defined and are bijective ones. This implies that all subspaces $V_{s_1,\ldots,s_k}$ are mutually isomorphic. The Theorem is proved.

We note that projector operators (3.13) with $\Gamma_{[i]}$ being a product of Γ-matrices were considered previously in [36].

4 Examples of supersymmetric solutions

4.1 $M2$-brane.

We consider the electric 2-brane solution defined on the manifold

$$M_0 \times M_1 \times M_2. \tag{4.1}$$

The solution reads

$$g = H^{1/3} \{ \hat{g}^0 + H^{-1} \hat{g}^1 + \hat{g}^2 \}, \tag{4.2}$$

$$F = \nu dH^{-1} \wedge \hat{\tau}_1, \tag{4.3}$$

where $\nu^2 = 1/2$, $H = H(x)$ is a harmonic function on (M_0, g^0) $d_1 = 3$, $d_0 + d_2 = 8$, and the metrics g^i, $i = 0, 1, 2$, are Ricci-flat.

4.1.1 Flat g^i

Let us consider a special case of flat g^i

$$g^0_{\mu\nu} = \delta_{\mu\nu}, \quad g^1_{m_1n_1} = \eta^{(1)}_{m_1n_1}, \quad g^2_{m_2n_2} = \delta_{m_2n_2} \tag{4.4}$$

where $(\eta^{(1)}_{a_1b_1}) = \mathrm{diag}(-1,+1,+1)$. We fix the frames in (2.9) as follows

$$e^{(0)a}{}_{\mu} = \delta^a_{\nu}, \qquad e^{(i)a_i}{}_{m_i} = \delta^{a_i}_{m_i}, \tag{4.5}$$

$i = 1,2$.

It may be verified using relations from subsections 2.3 and 2.4 and formulae (4.4) and (4.5) that the SUSY eqs. (3.2) are satisfied identically if

$$\varepsilon = H^{-1/6}\varepsilon_*, \qquad \varepsilon_* = \mathrm{const}, \tag{4.6}$$

$$\Gamma\varepsilon_* = c\varepsilon_*, \qquad c = \mathrm{sign}\nu, \tag{4.7}$$

where

$$\Gamma = \hat{\Gamma}^{1_1}\hat{\Gamma}^{2_1}\hat{\Gamma}^{3_1}. \tag{4.8}$$

Here Γ is real-valued matrix satisfying $\Gamma^2 = \mathbf{1}$, where $\mathbf{1}$ is unit 32×32-matrix. Let $A = \hat{\Gamma}^{1_0}$. The pair $\Gamma = \Gamma_{[1]}, A = A_{[1]}$ satisfies the conditions of the **Theorem**, and hence for $\varepsilon_1 \in \mathbf{R}^{32}$ the dimension of the subspace V_c of solutions to eqs. (4.7) is 16. For $\varepsilon_1 \in \mathbf{G}^{32}_1$ (see **Remark 1**) the (odd part of) superdimension of the subsuperspace $\mathbf{V_c}$ from (4.7) is also 16. This means that (at least) $N = 1/2$ part of SUSY is preserved.

4.1.2 Non-flat g^i.

Here we put $d_2 = 0$ in (4.1), i.e. we consider the metric on $M_0 \times M_1$:

$$g = H^{1/3}\{\hat{g}^0 + H^{-1}\hat{g}^1\}, \tag{4.9}$$

with $d_1 = 3$, $d_0 = 8$, and the form from (4.3) where metrics g^i, $i = 0,1$ are Ricci-flat; g^0 has Euclidean signature and g^1 has the signature diag$(-1,+1,+1)$.

Let us consider Γ-matrices

$$(\hat{\Gamma}^A) = (\hat{\Gamma}^a_{(0)} \otimes \mathbf{1}_2, \hat{\Gamma}_{(0)} \otimes \hat{\Gamma}^{a_1}_{(1)}), \tag{4.10}$$

where $\hat{\Gamma}^a_{(0)}$, $a = 1_0,\dots,8_0$ correspond to M_0 and $\hat{\Gamma}^{a_1}_{(1)}$, $a_1 = 1_1, 2_1, 3_1$ correspond to M_1 and $\hat{\Gamma}_{(0)} = \hat{\Gamma}^{1_0}_{(0)} \dots \hat{\Gamma}^{8_0}_{(0)}$. The substitution

$$\varepsilon = H^{-1/6}\eta_0(x) \otimes \eta_1(y), \tag{4.11}$$

$$\Gamma\varepsilon = c_\nu\varepsilon, \qquad c = \mathrm{sign}\nu, \tag{4.12}$$

where Γ is defined in (4.8), $\eta_0(x)$ is a 2-component Killing spinor on M_0 and $\eta_1(y)$ is a 16-component Killing spinor on M_1, i.e.

$$D^{(0)}_\mu \eta_0 = D^{(1)}_{m_1}\eta_1 = 0, \tag{4.13}$$

with $D^{(0)}_{\mu} = \partial_{\mu} + \frac{1}{4}\omega^{(0)}_{ab\mu}\hat{\Gamma}^{a}_{(0)}\hat{\Gamma}^{b}_{(0)}$ and $D^{(1)}_{m_1} = \partial_{m_1} + \frac{1}{4}\omega^{(1)}_{a_1 b_1 m_1}\hat{\Gamma}^{a_1}_{(1)}\hat{\Gamma}^{b_1}_{(1)}$, gives us a solution to the SUSY equations.

We get from (4.10) that

$$\Gamma = \hat{\Gamma}_{(0)} \otimes \hat{\Gamma}_{(1)}, \tag{4.14}$$

where $\hat{\Gamma}_{(1)} = \hat{\Gamma}^{1_1}_{(1)}\hat{\Gamma}^{2_1}_{(1)}\hat{\Gamma}^{3_1}_{(1)}$. Choosing real matrices $\hat{\Gamma}^{1_1}_{(1)} = i\sigma_2$, $\hat{\Gamma}^{2_1}_{(1)} = \sigma_1$, $\hat{\Gamma}^{3_1}_{(1)} = \sigma_3$, (where σ_i are the standard Pauli matrices) we get $\hat{\Gamma}_{(1)} = \mathbf{1}_2$, and due to eq. (4.14) the relation (4.12) is equivalent to the following

$$\hat{\Gamma}_{(0)}\eta_{(0)} = c\eta_{(0)}. \tag{4.15}$$

Hence the number of unbroken SUSY is (at least)

$$N = n_0(c)n_1/32, \tag{4.16}$$

where $n_0(c)$ is the number of chiral Killing spinors on M_0 satisfying (4.15) with $c = \mathrm{sign}\nu$, and n_1 is the number of Killing spinors on M_1.

4.2 $M5$-brane

Now let us consider the magnetic 5-brane solution defined on the manifold (4.1),

$$g = H^{2/3}\{\hat{g}^0 + H^{-1}\hat{g}^1 + \hat{g}^2\}, \tag{4.17}$$

$$F = \nu(*_0 dH) \wedge \hat{\tau}_2, \tag{4.18}$$

where $\nu^2 = 1/2$, $H = H(x)$ is a harmonic function on (M_0, g^0), $d_1 = 6$, $d_0 + d_2 = 5$ and metrics g^i, $i = 0, 1, 2$, are Ricci-flat.

4.2.1 Flat g^i

Let all metrics be flat, i.e. we consider the relations (4.4) with $(\eta^{(1)}_{a_1 b_1}) = \mathrm{diag}(-1, +1, +1, +1, +1, +1)$. We also consider canonical frames defined by (4.5).

The SUSY eqs. (3.2) are satisfied identically if

$$\varepsilon = H^{-1/12}\varepsilon_*, \qquad \varepsilon_* = \mathrm{const}, \tag{4.19}$$

where ε_* obeys to eq. (4.7) with

$$\Gamma = \hat{\Gamma}^{\bar{1}}\hat{\Gamma}^{\bar{2}}\hat{\Gamma}^{\bar{3}}\hat{\Gamma}^{\bar{4}}\hat{\Gamma}^{\bar{5}} \tag{4.20}$$

and $(\bar{1}, \ldots, \bar{5}) = (1_0, \ldots, (d_0)_0, 1_2, \ldots, (d_2)_2)$. Here $\Gamma^2 = \mathbf{1}$. Let $A = \hat{\Gamma}^{1_1}$. The pair $\Gamma = \Gamma_{[1]}, A = A_{[1]}$ satisfies the conditions of the **Theorem**. Hence we obtain that $N = 1/2$ part of supersymmetries is preserved.

4.2.2 Non-flat g^i.

Let $d_2 = 0$ in (4.17), i.e. we consider the metric on $M_0 \times M_1$

$$g = H^{2/3}\{\hat{g}^0 + H^{-1}\hat{g}^1\}, \tag{4.21}$$

with $d_1 = 6$, $d_0 = 5$, where metrics g^i, $i = 0, 1$, are Ricci-flat, g^0 has a Euclidean signature and g^1 has the signature diag$(-1, +1, +1, +1, +1, +1)$. The 4-form (4.18) is modified as follows

$$F = \nu(*_0 dH). \tag{4.22}$$

Let us consider Γ-matrices

$$(\hat{\Gamma}^A) = (\hat{\Gamma}^a_{(0)} \otimes \hat{\Gamma}_{(1)}, \mathbf{1}_4 \otimes \hat{\Gamma}^{a_1}_{(1)}), \tag{4.23}$$

where $\hat{\Gamma}^a_{(0)}$, $a = 1_0, \ldots, 5_0$, correspond to M_0 and $\hat{\Gamma}^{a_1}_{(1)}$, $a_1 = 1_1, \ldots, 6_1$, correspond to M_1 and $\hat{\Gamma}_{(1)} = \hat{\Gamma}^{1_1}_{(1)} \ldots \hat{\Gamma}^{6_1}_{(1)}$. The substitution

$$\varepsilon = H^{-1/12}\eta_0(x) \otimes \eta_1(y), \tag{4.24}$$

$$\Gamma\varepsilon = c_\nu \varepsilon, \qquad c = \mathrm{sign}\nu, \tag{4.25}$$

where Γ from (4.20) reads

$$\Gamma = \hat{\Gamma}^{1_0} \ldots \hat{\Gamma}^{5_0} = \hat{\Gamma}_{(0)} \otimes \hat{\Gamma}_{(1)}, \tag{4.26}$$

with $\hat{\Gamma}_{(0)} = \hat{\Gamma}^{1_0}_{(0)} \ldots \hat{\Gamma}^{5_0}_{(0)}$ gives us a solution to the SUSY equations. In (4.24) $\eta_0(x)$ is 4-component Killing spinor on M_0 and $\eta_1(y)$ is 8-component Killing spinor on M_1, i.e. the relations (4.13) are satisfied. We put $\hat{\Gamma}^{5_0}_{(0)} = \hat{\Gamma}^{1_0}_{(0)} \ldots \hat{\Gamma}^{4_0}_{(0)}$. Then $\hat{\Gamma}_{(0)} = \mathbf{1}_4$, and due to (4.26) the relation (4.25) reads

$$\hat{\Gamma}_{(1)}\eta_{(1)} = c\eta_{(1)}. \tag{4.27}$$

Hence, the number of preserved SUSY is (at least)

$$N = n_0 n_1(c)/32, \tag{4.28}$$

where $n_1(c)$ is the number of (chiral) Killing spinors on M_1 satisfying (4.27) with $c = \mathrm{sign}\nu$, and n_0 is the number of Killing spinors on M_0. A special case of this supersymmetric solution with $M_0 = \mathbf{R}^5$ and $M_1 = \mathbf{R}^2 \times K3$, was considered in [20]. In this case $N = 1/4$ in agreement with (4.28) since $n_0 = 4$ and (as can be easily verified) $n_1(c) = n[K3] = 2$ (i.e. the number of chiral Killing spinors on $\mathbf{R}^2 \times K3$ is equal to the total number of Killing spinors on $K3$). We remind that $K3$ is a 4-dimensional Ricci-flat Kähler manifold with $SU(2)$ holonomy group and self-dual (or anti-self-dual) curvature tensor. $K3$ has two Killing spinors (left or right).

4.3 $M2 \cap M5$-branes

Here we consider solutions with two "orthogonally" intersecting p-branes (with $p = 2, 5$) defined on the manifold

$$M_0 \times M_1 \times M_2 \times M_3 \times M_4 \tag{4.29}$$

to show how the Theorem works.

The solution with $M2$ and $M5$ branes defined on the manifold (4.29) reads [31, 32, 33, 2]

$$g = H_1^{1/3} H_2^{2/3} \{\hat{g}^0 + H_1^{-1}\hat{g}^1 + H_2^{-1}\hat{g}^2 + H_1^{-1}H_2^{-1}\hat{g}^3 + \hat{g}^4\}, \tag{4.30}$$

$$F = \nu_1 dH_1^{-1} \wedge \hat{\tau}_1 \wedge \hat{\tau}_3 + \nu_2 (*_0 dH_2) \wedge \hat{\tau}_1 \wedge \hat{\tau}_4, \tag{4.31}$$

where $\nu_1^2 = \nu_2^2 = 1/2$; H_1, H_2 are harmonic functions on (M_0, g^0), $d_1 = 1$, $d_2 = 4$, $d_3 = 2$, $d_0 + d_4 = 4$, and metrics g^i, $i = 0, 1, 2, 3, 4$, are Ricci-flat.

Let all g^i be flat:

$$g^0_{\mu\nu} = \delta_{\mu\nu}, \quad g^3_{m_1 n_1} = \eta^{(3)}_{m_1 n_1}, \quad g^i_{m_i n_i} = \delta_{m_i n_i}, \quad i = 1, 2, 4, \tag{4.32}$$

where $(\eta^{(3)}_{a_1 b_1}) = \mathrm{diag}(-1, +1)$. We consider the frames from (4.5) with $i = 1, 2, 3, 4$.

The SUSY eqs. (3.2) are satisfied identically if

$$\varepsilon = H_1^{-1/6} H_2^{-1/12} \varepsilon_*, \qquad \varepsilon_* = \mathrm{const}, \tag{4.33}$$

$$\Gamma_{[i]} \varepsilon_* = c_i \varepsilon_*, \qquad c_i = \mathrm{sign}\nu_i, \tag{4.34}$$

$i = 1, 2$, where

$$\Gamma_{[1]} = \hat{\Gamma}^{1_1} \hat{\Gamma}^{1_3} \hat{\Gamma}^{2_3}, \qquad \Gamma_{[2]} = \hat{\Gamma}^{\bar{1}} \hat{\Gamma}^{\bar{2}} \hat{\Gamma}^{\bar{3}} \hat{\Gamma}^{\bar{4}} \hat{\Gamma}^{\bar{5}}, \tag{4.35}$$

$(\bar{1}, \ldots, \bar{5}) = (1_0, \ldots, (d_0)_0, 1_1, 1_4, \ldots, (d_4)_4)$.

Introducing the "complimentary" matrices $A_{[1]} = \hat{\Gamma}^{1_0}$ and $A_{[2]} = \hat{\Gamma}^{1_3}$, we get from the **Theorem** that the (super)dimension of the (super)subspace of solutions to eqs. (4.34) is 8, i.e. at least $N = 1/4$ part of SUSY "survives".

Remark 2. The configurations under consideration remain supersymmetric if the functions H_i are arbitrary (not obviously harmonic ones). Thus, we are led to supersymmetric field sets that are not solutions to the equations of motion.

5 Conclusions

Thus here we considered the "Killing-like" SUSY equations for $D = 11$ supergravity in the backgrounds with a block-diagonal metric defined on the product of Ricci-flat spaces $M_0 \times M_1 \times \ldots \times M_n$.

We obtained decomposition relations for ingredients of the "Killing-like" SUSY equations (e.g. spin connection, matrix-valued covector field) and proved the 2^{-k}-splitting theorem for k commuting linear operators. We considered examples of $M2$ and $M5$

branes defined on the product of two Ricci-flat spaces and obtained formulae for a fractional number of unbroken SUSY. Also p-brane $M2 \cap M5$-configuration defined on the product of four flat spaces is considered to illustrate how the "2^{-k}-splitting" theorem works.

Other examples (with several branes on the products of Ricci-flat spaces) in different supergravity models will be considered in a separate publications.

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